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# Energy Spectrum of the Two-Dimensional $q$-Hydrogen Atom

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The discrete energy spectrum of the $q$-analog of the two-dimensional hydrogen atom is derived by deforming the Pauli equation. It contracts to that of the ordinary two-dimensional hydrogen atom in the limit $q \to \pm 1$. The degeneracy is discussed generally and some properties of the $q$-energy spectrum are studied both for real $q$ and for complex $q$ of magnitude unity.

The quantum group, the deformation of Lie algebra, has been studied by many researchers (Drinfel'd, 1985; Jimbo, 1985; Woronowicz, 1987). In order to explore the application of the quantum group, some $q$-analogs of dynamical systems have been discussed. Biedenharn (1989), Macfarlane (1989), and Sun and Fu (1989) realized the quantum group $SU_q(2)$ in terms of the $q$-analog of the harmonic oscillator. Kibler and Negadi (1991) gave the $q$-analog of the three-dimensional hydrogen atom. Yang and Xu (1993) also researched the $q$-analog of the three-dimensional hydrogen atom by applying the Kastaanheimo–Stiefel transformation and the $q$-oscillator. Chan and Finkelstein (1994) deformed the hydrogen atom from the point of view of the wave function in the group space of $SO(3)$.

In this paper, the $su(2)$ symmetry in the ordinary two-dimensional hydrogen atom will be deformed to the quantum group $SU_q(2)$. Its discrete energy spectrum will be derived, its degeneracy will be discussed, and some properties of the energy spectrum will be studied both for real and unitary ($|q|^2 = 1$) quantum deformation parameter.

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It is well known that there is a Lie algebra $su(2)$ [or $so(3)$] in the two-dimensional hydrogen atom. It is generated by

$$J_1 = \left(-\frac{\mu e^4}{2E}\right)^{1/2} \frac{A_x}{\hbar}, \qquad J_2 = \left(-\frac{\mu e^4}{2E}\right)^{1/2} \frac{A_y}{\hbar}, \qquad J_3 = \frac{L}{\hbar} \tag{1}$$

where $E$ is the energy of the electron of the two-dimensional hydrogen atom, and

$$\begin{aligned} A_x &= \frac{1}{\mu e^2} L p_y + \frac{i\hbar}{2\mu e^2} p_x - \frac{x}{\rho} \\ A_y &= -\frac{1}{\mu e^2} L p_x + \frac{i\hbar}{2\mu e^2} p_y - \frac{y}{\rho} \end{aligned} \tag{2}$$

are the components of the conserved Runge-Lenz vector, and $L$ is the conserved orbital angular momentum. The generators (1) obey the commutation relation

$$[J_a, J_b] = i\epsilon_{abc} J_c$$

where $\epsilon_{abc}$ $(a, b, c = 1, 2, 3)$ stands for the permutation symbol. The Casimir operator of this $su(2)$ Lie algebra can be expressed as

$$J^2 = J_1^2 + J_2^2 + J_3^2 = -\frac{\mu e^4}{2E} - \frac{\hbar^2}{4} \tag{3}$$

Sometimes equation (3) will be called the Pauli equation, as for the ordinary three-dimensional hydrogen atom (Pauli, 1926).

A $q$-analog of the two-dimensional hydrogen atom can be realized by deforming the Lie algebra $su(2)$ to the quantum group $SU_q(2)$ that is described by the $q$-generators $J_{q+}$, $J_{q-}$, and $J_{q3}$, which obey the commutation relations

$$[J_{q3}, J_{q\pm}] = \pm J_{q\pm}, \qquad [J_{q+}, J_{q-}] = [2J_{q3}]_q \tag{4}$$

where

$$[x]_q = \frac{\sinh \eta x}{\sinh \eta}, \qquad \eta = \ln q$$

and the $q$-Casimir operator is

$$J_q^2 = J_{q-} J_{q+} + [J_{q3}]_q [J_{q3} + 1]_q \tag{5}$$

Correspondingly, from (1), one can define the $q$-angular momentum $L_q$ and the $q$-Runge-Lenz vector $\mathbf{A}_q$:

$$\mathbf{A}_q = A_{qx}\mathbf{i} + A_{qy}\mathbf{j}, \qquad L_q = \hbar J_{q3} \tag{6}$$

where

$$A_{qx} = \frac{1}{2}(A_{q+} + A_{q-}), \qquad A_{qy} = \frac{1}{2i}(A_{q+} - A_{q-})$$

and

$$A_{q\pm} = \hbar\left(\frac{2E_q}{\mu e^4}\right)^{1/2} J_{q\pm} \tag{7}$$

in which $E_q$ stands for the energy of the two-dimensional $q$-hydrogen atom. We have shown that the above-mentioned $q$-generators can be expressed by the generator of the Lie algebra (Zhang and Duan, 1994).

Using the Casimir operator (5), one can determine the $q$-analog of the Pauli equation (3) as follows:

$$J_q^2 = J_{q-}J_{q+} + [J_{q3}]_q[J_{q3} + 1]_q = -\frac{\mu e^4}{2E_q} - \frac{\hbar^2}{4} \tag{8}$$

The $q$-Pauli equation (8) reduces to the Pauli equation (3) of the ordinary two-dimensional hydrogen atom in the limit $q \to 1$. The energy spectrum of the $q$-analog of the two-dimensional hydrogen atom can also be determined from the $q$-Pauli equation (8) directly.

Let us consider the Hilbert space of the representation of the ordinary Lie algebra $su(2)$ as

$$H = \{\,|jm\rangle : j \in N;\, m = -j, -j + 1, \ldots, j\} \tag{9}$$

Because the generator $J_3$ in the two-dimensional hydrogen atom corresponds to the orbital angular momentum through (1), the indexes $j$ and $m$ in the Hilbert space (9) take integer values. Using the Jimbo representation of the quantum group $SU_q(2)$, we have

$$J_{q\pm}|jm\rangle = \{[j \mp m]_q[j \pm m + 1]_q\}^{1/2}|jm \pm 1\rangle$$

$$J_{q3}|jm\rangle = m$$

Then, acting with the $q$-Pauli equation (8) on the Hilbert space (9), we derive the energy spectrum of the $q$-analog of the two-dimensional hydrogen atom in the form

$$E_q = E_{qj} = -\frac{\mu e^4}{2\hbar^2}\frac{1}{[j]_q[j + 1]_q + 1/4} \tag{10}$$

When the quantum deformation parameter $q \to 1$, the energy spectrum (10) contracts to

$$E = E_s = -\frac{\mu e^4}{2\hbar^2 s^2}$$

with

$$s = j + \frac{1}{2} = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$$

This is just the energy spectrum of the usual two-dimensional hydrogen atom.

For fixed $j$, the $q$-angular momentum $L_q$ may have $(2j + 1)$ eigenvalues that correspond to $(2j + 1)$ states. Thus, the degeneracy of the energy spectrum (10) is $(2j + 1)$. This is the same as the energy spectrum of the ordinary two-dimensional hydrogen atom.

It is obvious that the ground-state level $(j = 0)$ of the $q$-analog of the two-dimensional hydrogen atom is equal to that of the ordinary two-dimensional hydrogen atom. Both are nondegenerate in case of real $q$.

Generally speaking,

$$[j]_q \geq j$$

for positive real $q$. From equation (10), one can easily deduce

$$E_q \geq E$$

in the above-mentioned case. In other words, the $q$-deformation of the Pauli equation makes the energy of the two-dimensional hydrogen atom higher for positive real quantum deformation parameter.

When the $q$-deformation parameter is a phase ($q = e^{i\eta}$ with real $\eta$), the energy spectrum is very interesting. It varies periodically with $\eta$. In the limit $\eta \to n\pi$ ($n$ = integer), i.e., $q \to \pm 1$, the energy spectrum (10) also reduces to that of ordinary two-dimensional hydrogen atom. If $\eta = \pi/n$, the energy $E_{qj}$ takes the value of the ground state of the $q$-analog of the two-dimensional hydrogen atom in the case where $j/n$ or $(j + 1)/n$ is an integer. The energy spectrum (10) also is distributed periodically within quantum number $j$. But there are only $n$ energy levels in this case. The degeneracy of the energy $E_{qj}$ becomes complicated. Sometimes it is infinite.

In this paper, a $q$-analog of the two-dimensional hydrogen atom has been proposed through deforming the Lie algebra $su(2)$ in the system into the quantum group $SU_q(2)$. The corresponding discrete energy spectrum was obtained in (10). It was shown that the $q$-energy spectrum reduces to that of the ordinary two-dimensional hydrogen atom in the limit $q \to \pm 1$. Generally, the degeneracy of the $q$-energy level is the same as that of the ordinary system. The energy level is deformed to be higher for positive real $q$. When $q$ is a phase, the $q$-energy spectrum becomes more interesting, in that the total number of energy levels can be determined by the quantum deformation parameter as $n = (i \ln q)/\pi$.

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